Field Paper:
Monopoly Pricing of Durable Goods and their Accessories

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1 Introduction

A durable good is one that retains its value in use through multiple periods of consumption. A significant proportion of durable goods, including appliances, video games, smartphones, and many types of toys, are complex products that come equipped with a standard set of features. Firms that sell those products also tend to sell a variety of accessories that provide extra utility to users by enhancing either the breadth or depth of the original goods' functions. Since an accessory often relies on the original product in order to function properly, one might guess that it would be priced to attract only the uppermost segment of consumers.

For the purposes of this paper, I treat an accessory as any good that is strictly meant to enhance another one. The model that I provide addresses the following question: How might a monopolistic firm price an original good and its accessory when the utility of the latter relies completely on the former? Through various modifications, the model will also address how the firm might price those goods when it can also offer a bundle of them, or when it can track its customers according to their purchasing histories and thereby charge them different prices in the future.

The model undertakes a two-period, bare bones approach to analyzing accessory pricing. Consumers have private types that determine how much utility they gain from consuming an original good and its complimentary accessory. Consumers' types are one-dimensional, thus implying perfect correlation between their enjoyments of the original and the accessory relative to other consumers. In the first period, the monopolist posts a price for the original good and consumers choose either to buy or pass. In the second period, the firm offers a new price for the original as well as an upgrade price, and the consumers again have the choice of buying or passing according to their desired combinations.

In contrast to some models, decisions about which or how many products to sell in a given time period are not left up to the firm. In the case of commitment, the firm may functionally preclude consumers from buying the original in a given period, but in general that is not the case. It may seem arbitrary to only allow the accessory to be sold in period 2, but some brief reflection should show that this is not altogether unrealistic. In many of the industries where firms sell durable goods and their accessories, a firm will not have even developed all of the accessories to a certain good by the time that it releases it. For instance, Hasbro only released the Barbie Dreamhouse once Barbie had proven successful in the marketplace.

2 Model Setup

Suppose that there is a monopolist that is selling two goods over the course of two periods. In the first period, the firm offers to sell the "original" good, which is durable and of quality $\theta_o > 0$, at a price of p_1. In the second period, the firm posts a new price for the original, p_2, while offering to sell an accessory of quality $\theta_a > 0$ at price p_a. In later versions, the firm may also have the option of selling a bundle of the two goods at a price of p_b in the second period, or it may choose to discriminate with respect to its second period prices based on whether consumers have bought the original already. This latter scenario is what I call "tracking," and it is assumed to be costless.

On the demand side of the market, there exists a unit measure of consumers, whose types are continuous random variables sharing cumulative distribution function $F(\cdot)$. Consumers' types are denoted by $\lambda \in [0, 1]$. To avoid mass points and gaps in the distribution of consumers, assume that F is strictly increasing and differentiable, and denote its corresponding p.d.f. by f.

Given the construction above, a consumer's type may be interpreted as the fraction of the potential quality that she utilizes when consuming each good. The obvious drawback of the typespace is that in being one-dimensional, it implies that the consumers' predilections for the original good and its accessory are perfectly correlated. Nonetheless, that might not be such a problem, since consumers who value the original more are intuitively more likely to value an accompanying accessory. Dimensional simplifications tend to expedite the analysis considerably as well.

The instantaneous utility garnered by a consumer of type λ who owns a good of quality θ is defined simply as

$$\lambda\theta.$$

Since the accessory is assumed to exhibit perfect complimentarity with the original, a consumer who buys the accessory without owning the original gains no instantaneous utility from doing so. Meanwhile, if a consumer has bought the good at a price of p in a certain period, her instantaneous surplus in that same period is defined as

$$\lambda\theta - p.$$

The basic feature of a durable good is that it can be consumed for multiple periods beyond the period that it was purchased. Therefore, any consumer who purchases the original good early will gain utility from it in both periods, but must only pay for it once.

I assume that each consumer has unitary absolute demand for each good. The effective demand for a given good is therefore equal to the measure of consumers who have chosen to buy it at its posted price. The firm's ex post profit from selling a given good is

$$Q(p)(p - c),$$

where $Q(p)$ is the measure of consumers who bought it given the endogenous price vector p and c is its constant marginal cost of provision. Fixed costs are assumed to be zero.

As I will demonstrate, the consumers' optimal strategies will be determined by cutoffs. Therefore, no significant loss of generality accrues from normalizing both goods' marginal costs to zero. The benefit of this assumption is that it provides an easy benchmark from which to judge the magnitude of the monopolist's rents. If all costs are normalized to zero, then the surplus-maximizing, perfectly competitive outcome occurs when every firm sets its prices equal to zero and every consumer buys the original good in period 1 and the accessory in period 2. Any cost normalization of this sort also allows for a reinterpretation of a good's quality as its corresponding maximum possible marginal instantaneous surplus from trade.

The present model makes use of a few other assumptions besides those mentioned above. For one, it assumes that there is no depreciation of the original good from the first period to the second. This implies that a given consumer will never has an incentive to buy the original both early and late. Secondly, it assumes that consumers and the firm each discount their second period surpluses at a rate of $\beta \in (0, 1]$, which is standard. Finally, it assumes that each consumer's type is privately known, while the rest of the agents only know the consumers' type distributions. Given that the monopolist is assumed to be risk-neutral, its objective is thus to maximize its expected profit under incomplete information.

As the analysis moves from one version of the model to the next, the strategy spaces for consumers and the firm will change accordingly. In some cases, the firm will be able to choose a price for the bundle, while in others, it will utilize its ability to track. In the latter cases, I will ignore the potential for secondhand markets, thereby precluding the possibility for arbitrage by consumers who are offered the upgrade at the lowest price. This assumption seems reasonable in real-world scenarios where the costs of sustaining such markets is prohibitively high.

The equilibrium concept most relevant to the model is that of Bayesian Nash equilibrium. Without knowing anybody's type in particular, the firm endeavors to maximize its expected profits given the distribution of consumers and given how it expects each consumer type to react to its pricing policy. The optimal strategies for consumers, meanwhile, will be contingent upon their individual types. When the firm is not bound by commitment, the best responses of all of the agents can be determined by backwards induction.

In order to provide examples of clean, closed form solutions for each version of the model, at times I will make the further assumption that the consumers' types are uniformly distributed. This will serve to simplify the first order conditions immensely, as $F(x)$ reduces to x and $f(x)$ reduces to 1 for any $x \in [0, 1]$ under the standard uniform distribution.

3 The Case of Commitment

In the preliminary version of the model, the monopolist commits to all of its prices in the first period. In the second period, it can neither bundle the two goods nor track consumers' purchases. Consequently, it can only segment its potential customers along three criteria: whether they buy the original, in what period they choose to buy the original, and whether they buy the accessory. The individual rationality (IR) and incentive compatibility (IC) constraints facing a utility-maximizing consumer of type λ are:

- IR on buying early: $\max\{(1+\beta)\lambda\theta_o - p_1 + \beta(\lambda\theta_a - p_a), (1+\beta)\lambda\theta_o - p_1\} \geq 0$, or equivalently,
$$\lambda \geq \min\left\{\frac{p_1 + \beta p_a}{(1+\beta)\theta_o + \beta\theta_a}, \frac{p_1}{(1+\beta)\theta_o}\right\}; \tag{1}$$
- IR on buying late: $\max\{\beta(\lambda\theta_o - p_2) + \beta(\lambda\theta_a - p_a), \beta(\lambda\theta_o - p_2)\} \geq 0$, or equivalently,
$$\lambda \geq \min\left\{\frac{p_2 + p_a}{\theta_o + \theta_a}, \frac{p_2}{\theta_o}\right\}; \tag{2}$$
- IC on buying early rather than late: $(1+\beta)\lambda\theta_o - p_1 \geq \beta(\lambda\theta_o - p_2)$, or equivalently,
$$\lambda \geq \frac{p_1 - \beta p_2}{\theta_o}; \tag{3}$$
- IR on buying the accessory: $\beta(\lambda\theta_a - p_a) \geq 0$, or equivalently,
$$\lambda \geq \frac{p_a}{\theta_a}. \tag{4}$$

The reason why the consumer gains utility $(1+\beta)\lambda\theta_o$ from buying the original early is because she gets to enjoy the good not just in the first period, but also in the second. Depreciation does not occur, so the utility from buying early can unequivocally be compounded in this way.

The individual rationality constraints on buying the original in either period contain not just one, but two terms. Each one must capture the possibility that some consumers might only buy the original at all so that they can enjoy the accessory as well. If consumers foresee large surpluses from buying the accessory, then they may be induced to buy the original even if they would lose surplus from buying it solitarily. As I will show later, this never actually happens.

The above constraints are enough to show that the consumers' optimal strategies will be determined by cutoffs that are functions of the firm's posted prices. Before setting up the firm's maximization problem, the order of the consumers' cutoffs, as well as an apprehension of which ones constitute the relevant constraints on buying the original, must be determined. The following claim helps to narrow down the possibilities:

Claim 1. *Whenever the monopolist is bound by commitment, all Bayesian Nash equilibria include a strictly positive measure of consumers whose types range from some cutoff to 1 who buy the original early.*

Proof. Suppose not. Then any consumer who buys the original at all would rather buy it late (implying that the incentive compatibility constraint must be greater than or equal to 1). Moreover, all of them will have types greater than or equal to the relevant IR constraint on buying late. However, the outcome of this supposed equilibrium is clearly suboptimal for the monopolist, which can make strictly more profit for itself and garner more surplus for each of its present customers by reducing p_1 to the point that all of them now prefer to buy the original early instead of late.

Indeed, in the case with no bundling or tracking, and where nobody is buying the original just for the sake of the accessory ($IR(late) = \frac{p_2}{\theta_o}$), the best possible deviation is for the firm to set $p_1' = (1 + \beta)p_2$ and $p_2' = p_2$ (where the primes indicate the prices of the original good after deviation and p_2 is the postulated equilibrium price of the original good in period two). By replacing the old IR constraint on buying late with a new IC constraint on buying early, the monopolist can effectively persuade consumers to buy early instead of late. The result is a gain of $\lambda\theta_o - p_2$ in present value surplus for each of the already-purchasing consumers (keeping in mind that all of them already had types satisfying $\lambda\theta_o \geq p_2$) and an increase in present value profit for the firm of $[1 - F(\frac{p_2}{\theta_o})]p_2$.

Analogous arguments may be applied for Bayesian equilibria under commitment with either bundling or tracking. This becomes especially apparent upon noticing that the monopolist cannot profit from either of those options without selling the original to some of its customers early in the first place. Otherwise, it would have no means by which to differentiate according to consumers' purchases.

Finally, and to reiterate, it is clear that the consumers who buy the original early will do so according to a cutoff, since increasing their types will strictly increase the surplus that they can gain by purchasing, and at a faster rate than if they were buying late to boot! □

In terms of the consumers' cutoffs, Claim 1 implies that $IR(late) \leq IC < 1$. Nonetheless, this still leaves open a variety of possible cutoff configurations. For instance, when equilibrium is achieved, will any consumers buy the original late? Are there any circumstances where some consumers might want to buy the original just so that they can gain even more back when they buy the accessory? It turns out that the answer to each of those questions is "no."

No matter how you organize the cutoffs, the firm's first order conditions always return the same outcome: all of the consumers who buy the original good buy it early, and they also all end up buying the accessory. The equilibrium cutoff for the consumers who buy is equal to the fixed point of the inverse hazard rate of the cumulative distribution function, conventionally defined by[1]

$$h^{-1}(\lambda_f) \equiv \frac{1 - F(\lambda_f)}{f(\lambda_f)}.$$

Supposing that the firm still makes the vain offer of a price for the original good in period two, its pricing policy can be expressed by the rule

$$\frac{p_2}{\theta_o} = \frac{p_1 - \beta p_2}{\theta_o} = \frac{p_a}{\theta_a} = \lambda_f,$$

where λ_f is as defined above. The solution to the equations is readily apparent. The firm's resulting profit is equal to

$$\Pi_{commit} = \lambda_f[1 - F(\lambda_f)][(1+\beta)\theta_o + \beta\theta_a].$$

The conclusion that the monopolist will set its prices to avoid selling the original good in period two dovetails nicely with that of Stokey (1981), who demonstrated that a multi-period price-discriminating durable good monopolist maximizes its profit under commitment by reducing itself down to a single-period monopolist that only sells its good in the first period. In fact, since the decision to buy the accessory in the preliminary case turns out to be mathematically independent of the decisions of whether or when to buy the original good, the version without tracking or bundling is perfectly amenable to the framework that Stokey (1981) delineates.

The result is the same, even, for monopolists that can credibly commit to their prices while tracking or offering a bundle. In those cases, the way that a monopolist would utilize its discriminatory capabilities is by devising an incentive compatible means to differentiate

[1] For now, I will allow for distributions that admit a multiplicity of fixed points. When this is the case, the firm will choose to construct the consumer cutoffs at whatever fixed point yields the highest profits. This point will be the same for both the original good and the accessory since the preferred cutoff under commitment does not depend on the values of the goods' qualities.

between relatively high-typed and low-typed consumers. To successfully differentiate, the monopolist would have to sell the original good to some consumers early and to others late. Otherwise, it would have no way to glean any information about their underlying types before offering its prices for the accessory. Splitting up its sales of the original good causes the monopolist to sacrifice more profit on the original than it can ever hope to retrieve from differentiating with the accessory. Although they are messy, the first-order conditions corresponding to profit maximization under any of the cutoff configurations that might occur under a regime of tracking or bundling are sufficient to bear out the same conclusion.

Example 1. *(Uniformly distributed types) When consumers are uniformly distributed, a unique fixed point of the inverse hazard rate occurs at* $\lambda_f = \frac{1}{2}$*. In that instance, the top half consumer types buy each good, and the resulting profit is*

$$\Pi_{commit} = p_1 Q_1(p) + \beta p_a Q_a(p) = \frac{(1+\beta)\theta_o}{2}\left(1 - \frac{1}{2}\right) + \frac{\beta\theta_a}{2}\left(1 - \frac{1}{2}\right) = \frac{(1+\beta)\theta_o}{4} + \frac{\beta\theta_a}{4}.$$

4 Non-Commitment without Bundling or Tracking

Under non-commitment, the firm is not able to credibly reveal its second-period prices right away, but rather conditions them on its price choice from period 1. If consumers are to make informed decisions about purchasing the original good in period 1, then they must be able to anticipate the behavior of the monopolist in period 2. Consequently, the monopolist takes this into account when it chooses its prices.

Since the game is finite, a unique pure strategy equilibrium can be derived using backwards induction. In the second period, the firm knows that among the consumers who did not buy the original early, anyone whose type satisfies the relevant IR constraint on buying late will do so. It also knows the same for the accessory: any consumer whose type is above the relevant IR constraint will wish to buy. This raises the possibility of two different cutoff configurations. In the first, $\frac{p_2}{\theta_o} \leq \frac{p_a}{\theta_a}$, and any consumer who wants the accessory will have also gained exclusively from owning the original. In the second, though, $\frac{p_a}{\theta_a} < \frac{p_2}{\theta_o}$, and there will be some consumers who only buy the original for the priviledge of buying and consuming the accessory. In that case, the relevant IR constraints on each good are not $\frac{p_2}{\theta_o}$ and $\frac{p_a}{\theta_a}$, but are adjoined by the expression $\frac{p_2+p_a}{\theta_o+\theta_a}$, which lies strictly between them. As the subsequent claim shows, the second configuration is never strictly optimal.

Claim 2. *In the case of non-commitment without bundling or tracking, suppose that* $\frac{p_a}{\theta_a} < \frac{p_2}{\theta_o}$*, so that the shared IR constraint on buying the original late and buying the accessory is given* *by* $\frac{p_2+p_a}{\theta_o+\theta_a}$*. Then the firm can always do just as well for itself by resetting* p_2' *and* p_a' *so that* $\frac{p_a'}{\theta_a} = \frac{p_2'}{\theta_o} = \frac{p_2+p_a}{\theta_o+\theta_a}$.

Proof. Notice that under the new price configuration, nobody changes their subgame purchasing behavior. Any early buyers have already bought, while the IR constraints remain unchanged for everyone else. Given that p_a, p_a', p_2, and p_2' are all as defined above, let $\epsilon_a > 0$ and $\epsilon_2 > 0$ be such that $p_a' = p_a + \epsilon_a$ and $p_2' = p_2 - \epsilon_2$. Then $\frac{p_a'}{\theta_a} = \frac{p_a+\epsilon_a}{\theta_a} = \frac{p_2+p_a}{\theta_o+\theta_a}$ and $\frac{p_2'}{\theta_o} = \frac{p_2-\epsilon_2}{\theta_o} = \frac{p_2+p_a}{\theta_o+\theta_a}$ if and only if

$$\epsilon_a = \epsilon_2 = \frac{p_2\theta_a - p_a\theta_o}{\theta_o + \theta_a}.$$

To wit, the firm gains exactly as much by raising p_a as it loses by reducing p_2. This means that the firm can always be just as well off by choosing $\frac{p_2}{\theta_o} \leq \frac{p_a}{\theta_a}$ as $\frac{p_a}{\theta_a} < \frac{p_2}{\theta_o}$. □

Given that the cutoffs have been determined, the firm's second period optimization problem is to

$$\max_{p_2,p_a} \; [F(\gamma_1) - F(\frac{p_2}{\theta_o})]p_2 + [1 - F(\frac{p_a}{\theta_a})]p_a,$$

where γ_1 represents the lowest type that bought the original good in period 1. By the time the game reaches period 2, γ_1 has already been determined, so the monopolist treats it as given. The first order conditions corresponding to the firm's problem are:

[p_2]

$$(\frac{p_2}{\theta_o})f(\frac{p_2}{\theta_o}) + F(\frac{p_2}{\theta_o}) = F(\gamma_1);$$

[p_a]

$$(\frac{p_a}{\theta_a})f(\frac{p_a}{\theta_a}) + F(\frac{p_a}{\theta_a}) = 1.$$

There are two things to notice about the first order condition on p_a. First of all, it shows that p_a is not a function of γ_1. That implies that the firm's first-period pricing decision affects neither the price of the accessory nor how much of it is sold. Secondly, it is the same first-order condition as in the case of commitment. Therefore, the price of the accessory and its corresponding cutoff will be the same in both cases.

Denote the optimal cutoff solutions described by these conditions by $\gamma_2^*(\gamma_1)$ and γ_a^* and their corresponding optimal prices by $p_2^*(\gamma_1)$ and p_a^*. Backtracking into period 1, the lowest type consumer who buys the original good early is the one who is indifferent between buying early and buying late. Given the definition of γ_1, the consumer's type should satisfy $\lambda = \gamma_1 = \frac{p_1 - \beta p_2^*(\gamma_1)}{\theta_o}$, yielding a solution $p_1^*(\gamma_1) = \gamma_1\theta_o + \beta p_2^*(\gamma_1)$ for the price of the original good in period 1 given the cutoff γ_1. Understanding this, the firm chooses γ_1 to maximize its expected profit from all sales of the original. That is to say, it will solve the following problem:

$$\max_{\gamma_1} \ \Big[1 - F(\gamma_1)\Big]\Big[\gamma_1\theta_o + \beta p_2^*(\gamma_1)\Big] + \beta\Big[F(\gamma_1) - F\Big(\frac{p_2^*(\gamma_1)}{\theta_o}\Big)\Big]p_2^*(\gamma_1).$$

The optimality condition determining the firm's choice of a cutoff is derived by taking the first order condition of the above problem and simplifying it using the first order condition for $p_2^*(\gamma_1)$. The resulting optimality condition is

$$\frac{1 - F(\gamma_1)}{f(\gamma_1)} = \gamma_1\Big[\frac{\theta_o}{\theta_o + \beta p_2^{*\prime}(\gamma_1)}\Big],$$

where $p_2^{*\prime}(\gamma_1) \equiv \frac{d}{d\gamma_1}(p_2^*(\gamma_1))$. Denoting the solution to the above optimality condition by γ_1^* and inputting it into the relevant pricing equations yields the Bayesian Nash equilibrium pricing profile $\Big(p_1^*(\gamma_1^*), p_2^*(\gamma_1^*), p_a^*\Big)$, the members of which are all functions of the underlying parameters.

Example 2. *(Uniformly distributed types) When the consumers' types are uniformly distributed, $xf(x) + F(x) = 2x$, so working backwards from the firm's second period first order conditions yields $p_2^*(\gamma_1) = \frac{\gamma_1\theta_o}{2}$ and $p_a^* = \frac{\theta_a}{2}$. Using $p_2^{*\prime}(\gamma_1) = \frac{\theta_o}{2}$, the optimality condition determining the firm's cutoff choice is*

$$1 - \gamma_1 = \gamma_1\Big[\frac{\theta_o}{\theta_o + \beta(\frac{\theta_o}{2})}\Big].$$

Solving for the cutoff yields $\gamma_1^* = \frac{2+\beta}{4+\beta}$, from which we can compute $p_1^* = \frac{(2+\beta)^2\theta_o}{2(4+\beta)}$, $p_2^* = \frac{(2+\beta)\theta_o}{2(4+\beta)}$, and $\gamma_2^* = \frac{2+\beta}{2(4+\beta)}$. The firm's expected profit is therefore equal to

$$\begin{aligned}\Pi_{Non-commit} &= p_1Q_1(p) + \beta p_2Q_2(p) + \beta p_aQ_a(p)\\ &= \Big(\frac{\theta_o(2+\beta)^2}{2(4+\beta)}\Big)\Big(1 - \frac{2+\beta}{4+\beta}\Big) + \beta\Big(\frac{\theta_o(2+\beta)}{2(4+\beta)}\Big)\Big[\frac{2+\beta}{4+\beta} - \frac{2+\beta}{2(4+\beta)}\Big] + \frac{\beta\theta_a}{4}\\ &= \frac{\theta_o}{4}\Big[\frac{(2+\beta)^2}{4+\beta}\Big] + \frac{\beta\theta_a}{4}.\end{aligned}$$

Notice that when consumers are uniformly distributed and the firm cannot commit, it will end up selling less of the original good in the first period and more in the second period than under commitment. At this point, the discount factor begins to play a role. As β increases, γ_1^*, γ_2^*, p_1^*, and p_2^* are all increasing. Presumably, this is because the firm would like to cut into the extra potential surplus from trade in either period by restricting the supply of the good even further. Unfortunately, in the case on non-commitment, there is still no real interaction between the original good and the accessory. At this point, they may as well be two separate goods.

When consumers are uniformly distributed, $\Pi_{commit} \geq \Pi_{non-commit}\ \forall\ \beta \in (0, 1]$. The firm loses out when it is not able to commit credibly to all of its prices in the first period. If consumers apprehend that the firm will choose to sell the original good at a newly optimized price in the second period, then some of them will hold off purchasing the good until later. By offering a reasonable price for the original good in period two, the firm competes against itself. Taken to its logical limit, the manner of self-competition described here is what eventually leads to the eponymous "Coase Conjecture" developed by Coase (1972). Commitment, it turns out, is the only device that can shield the firm from its own self-defeating behavior.

5 Bundling and Tracking

As I have already stated, bundling and tracking offer the firm no advantages under a regime that allows commitment. Therefore, this section deals exclusively with non-commitment. The Bayesian Nash equilibria under tracking and bundling can be determined using the same backwards induction procedure as under basic non-commitment. Framing and comparing the firm's maximization problem in each case is sufficient to show that as far as the model is concerned the two devices are essentially the same. In both cases, I will define γ_1 to be the lowest-type consumer who buys the original good in period 1.

Starting with bundling, suppose that the monopolist can offer a bundle consisting of the original good and the accessory at a price of p_b in period 2. First of all, note that no consumer would ever buy the bundle unless p_b were less than $p_a + p_2$. Otherwise, they could gain more utility by simply purchasing the two goods separately. For now, assume that the condition holds; that it does hold will become apparent later.

The relevant constraints for consumers who did not buy the original early, i.e., whose types are in $[0, \gamma_1)$, are the following:

- IR on buying the original late: $\lambda\theta_o - p_2 \geq 0 \Leftrightarrow \lambda \geq \frac{p_2}{\theta_o}$;
- IR on buying the bundle: $\lambda(\theta_o + \theta_a) - p_b \geq 0 \Leftrightarrow \lambda \geq \frac{p_b}{\theta_o+\theta_a}$;
- IC on buying the bundle instead of the original: $\lambda(\theta_o+\theta_a)-p_b \geq \lambda\theta_o - p_2 \Leftrightarrow \lambda \geq \frac{p_b-p_2}{\theta_a}$.

Meanwhile, the only constraint for the early buyers, i.e. those whose types are in $[\gamma_1, 1]$, to consider is an

- IR on buying the accessory: $\lambda\theta_a - p_a \geq 0 \Leftrightarrow \lambda \geq \frac{p_a}{\theta_a}$.

Finally, since the firm only markets the accessory to the consumers who bought the original early, it will necessarily choose

$$\frac{p_a}{\theta_a} \geq \gamma_1.$$

Otherwise, it would needlessly forgo $\theta_a(\gamma_1 - \frac{p_a}{\theta_a})[1 - F(\gamma_1)]$ in potential second period profits, thereby committing an act that bears zero credibility within the second period subgame.

The firm's second period problem is to

$$\max_{p_a, p_b, p_2} \left[1 - F\Big(\frac{p_a}{\theta_a}\Big)\right]p_a + \left[F(\gamma_1) - F\Big(\frac{p_b - p_2}{\theta_a}\Big)\right]p_b + \left[F\Big(\frac{p_b - p_2}{\theta_a}\Big) - F\Big(\frac{p_2}{\theta_o}\Big)\right]p_2$$

subject to $\frac{p_a}{\theta_a} \geq \gamma_1$. Taking first order conditions and then simplifying leads to the following optimality conditions:

$$\frac{p_b - p_2}{\theta_a} = \frac{p_2}{\theta_o}; \quad F(\gamma_1) - F\Big(\frac{p_b - p_2}{\theta_a}\Big) = \Big(\frac{p_b - p_2}{\theta_a}\Big) f\Big(\frac{p_b - p_2}{\theta_a}\Big); \quad \frac{p_a}{\theta_a} = \max\{\gamma_1, \lambda_f\},$$

where λ_f is yet again the fixed point of the inverse hazard rate of F. An analogous result to that of Claim 2 applies to the case of bundling as well. Notice that the problem as currently outlined takes for granted the notion that the firm will want to choose $\frac{p_b - p_2}{\theta_a} \geq \frac{p_2}{\theta_o}$ so that the relevant constraints for the lower-type consumer are the IC on buying the bundle and the IR on buying the original. If instead it preferred to choose $\frac{p_b - p_2}{\theta_a} < \frac{p_2}{\theta_o}$, then its intention to never sell the original by itself would be explicit. Under that scenario, the only relevant constraint would be the IR on buying the bundle. But as first two of the optimality conditions show, both scenarios actually imply the same result. The first condition is sufficient to imply that $\frac{p_b}{\theta_o + \theta_a} = \frac{p_2}{\theta_o}$, while combining that with the second condition yields

$$F(\gamma_1) - F\Big(\frac{p_b}{\theta_o + \theta_a}\Big) = \Big(\frac{p_b}{\theta_o + \theta_a}\Big) f\Big(\frac{p_b}{\theta_o + \theta_a}\Big),$$

which is the exact optimality condition that arises from maximizing the firm's second period profit function under the latter scenario.

Using the optimality conditions just outlined, one can derive solutions $p_b^*(\gamma_1)$, $p_2^*(\gamma_1)$, and $p_a^* = \max\{\gamma_1\theta_a, \lambda_f\theta_a\}$, all of which are functions of γ_1. The contingent solution p_a^* introduces a dilemma for the firm when it endeavors to maximize in period 1. It is then that it must choose whether all or only some of the early buyers of the original will buy the accessory. If it chooses to sell to all of them, p_a^* will resolve to $\gamma_1\theta_a$, while if it wishes to sell to only some, p_a^* will equal $\lambda_f\theta_a$. In the latter case, the firm will also have to make sure to choose $\gamma_1 \leq \lambda_f$.

Incidently, the problem with tracking exhibits the exact same subgame characteristics as the one with bundling. To reiterate, tracking is the ability of the firm to costlessly condition its second period price offers on whether a given consumer has bought the original good early. Denoting $p_{a,1}$ as the accessory price offered to early buyers and $p_{a,2}$ as that offered to potential late buyers, and also applying the results of Claim 2 that the firm will never wish to set the IR on buying the accessory (for lower-type consumers) below the IR on buying the original good late, the relevant constraints for a consumer of type $\lambda \in [0, \gamma_1)$ are:

- IR on buying the original late: $\lambda\theta_o - p_2 \geq 0 \Leftrightarrow \lambda \geq \frac{p_2}{\theta_o}$;
- IR on buying the accessory: $\lambda\theta_a - p_{a,2} \geq 0 \Leftrightarrow \lambda \geq \frac{p_{a,2}}{\theta_a}$.

The relevant constraint for a consumer of type $\lambda \in [\gamma_1, 1]$ is simply

- IR on buying the accessory: $\lambda\theta_a - p_{a,1} \geq 0 \Leftrightarrow \lambda \geq \frac{p_{a,1}}{\theta_a}$.

Furthermore, since the monopolist is again only offering the price $p_{a,1}$ to early buyers, it will always choose

$$\frac{p_{a,1}}{\theta_a} \geq \gamma_1.$$

The firm's second period maximization problem is therefore to

$$\max_{p_2, p_{a,1}, p_{a,2}} \left[1 - F\left(\frac{p_{a,1}}{\theta_a}\right)\right]p_{a,1} + \left[F(\gamma_1) - F\left(\frac{p_2}{\theta_o}\right)\right]p_2 + \left[F(\gamma_1) - F\left(\frac{p_{a,2}}{\theta_a}\right)\right]p_{a,2}$$

subject to $\frac{p_{a,1}}{\theta_a} \geq \gamma_1$ and the trivial constraint $\frac{p_{a,2}}{\theta_a} \geq \frac{p_2}{\theta_o}$. The corresponding optimality conditions are thus given by

$$\frac{p_{a,2}}{\theta_a} = \frac{p_2}{\theta_o}; \quad F(\gamma_1) - F\left(\frac{p_2}{\theta_o}\right) = \left(\frac{p_2}{\theta_o}\right)f\left(\frac{p_2}{\theta_o}\right); \quad \frac{p_a}{\theta_a} = \max\{\gamma_1, \lambda_f\}.$$

Viewed alongside the optimality conditions for the firm under bundling, these conditions make it clear that the two devices are equivalent. In both cases, the firm sells the accessory to every late buyer of the original following a cutoff $\hat{\lambda}$ decided by the equation $F(\gamma_1) - F(\hat{\lambda}) = \hat{\lambda}f(\hat{\lambda})$. The optimality conditions for selling the accessory to higher-type consumers are identical, while both cases produce equivalent expected profit functions for the firm as induction moves the problem into period 1.

Conceptually, the functional equivalence of bundling and tracking makes sense as well, as each device grants the firm a way to split up the demand curve for the accessory by distinguishing between early and late buyers of the original good. The differentiation works, of course, because of the correlation between the consumers' valuations of the two goods. With bundling, the firm differentiates implicitly by tying the accessory to the original, which early buyers have no need for. With tracking, it differentiates quite explicitly, offering different prices to consumers based on their purchasing histories.

In practice, the two devices are likely to lose much of their parity. In many settings, bundling has two significant advantages over tracking. First of all, it is usually less costly and easier to enforce. Since the model assumes that both devices are costless and perfectly enforceable, neither of those qualities gets picked up. Bundling also allows the firm to price differentiate against its most eager consumers more covertly than tracking. Many consumers would feel betrayed by a company that decided to charge them extra for an accessory merely because they were more loyal or enthusiastic than their fellow consumers. But since bundling lumps the discounted price for the accessory along with a discounted price for the original, consumers are less likely to notice that they are being discriminated against when they pay the higher accessory price.

To finish off the problem, let's return to the case of bundling. The ambiguity in the solution for p_a^* implies that the firm must choose between two options: sell the accessory to all early buyers of the original or sell it to only some.

5.1 Case 1. All early buyers buy the accessory.

When the monopolists intends to induce every early buyer of the original to purchase the accessory, it will know in advance that $p_a^*(\gamma_1) = \gamma_1\theta_a$. The consumer who is indifferent between buying early and late will have the type γ_1 characterized by

$$(1+\beta)\gamma_1\theta_o - p_1 + \beta(\gamma_1\theta_a - p_a^*(\gamma_1)) = \beta[\gamma_1(\theta_o+\theta_a) - p_b^*(\gamma_1)] \Leftrightarrow \gamma_1 = \frac{p_1 - \beta p_b^*(\gamma_1)}{\theta_o - \beta\theta_a}.$$

Rearranging the equation and inserting $p_a^*(\gamma_1) = \gamma_1\theta_a$ yields

$$p_1^*(\gamma_1) = \gamma_1(\theta_o - \beta\theta_a) + \beta p_b^*(\gamma_1).$$

The firm's first period problem is therefore to

$$\max_{\gamma_1}\ [1-F(\gamma_1)][\gamma_1(\theta_o-\beta\theta_a)+\beta p_b^*(\gamma_1)]+\beta\Big[F(\gamma_1)-F\Big(\frac{p_b^*(\gamma_1)}{\theta_o+\theta_a}\Big)\Big]p_b^*(\gamma_1)+\beta[1-F(\gamma_1)]\gamma_a\theta_a \Rightarrow$$

$$\max_{\gamma_1}\ [1-F(\gamma_1)]\gamma_1\theta_o + \beta p_b^*(\gamma_1)\Big[1-F\Big(\frac{p_b^*(\gamma_1)}{\theta_o+\theta_a}\Big)\Big].$$

Taking the first order condition with respect to γ_1 and then simplifying it using the second period optimality conditions yields the following first period optimality condition:

$$\frac{1-F(\gamma_1)}{f(\gamma_1)} = \gamma_1\Big[\frac{\theta_o}{\theta_o + \beta p_b^{*\prime}(\gamma_1)}\Big],$$

where $p_b^{*\prime}(\gamma_1) \equiv \frac{d}{d\gamma_1}(p_b^*(\gamma_1))$. When the firm sells the accessory to every early buyer, the cutoff for late buyers is determined in much the same way with bundling or tracking as without them. The major difference between using the differentiation device and not using it is that the former allows the firm to split the buyers of the accessory along the same lines that it splits the buyers of the original. Solving for γ_1^* and then inserting it into the solution functions $p_1^*(\gamma_1)$, $p_2^*(\gamma_1)$, $p_b^*(\gamma_1)$, and $p_a^*(\gamma_a)$ yields the firm's full equilibrium pricing policy under bundling. Under tracking, the solution is the same, except that $p_{a,1}^* \equiv p_a^*$ and $p_{a,2}^* \equiv p_b^* - p_2^*$, both of which are merely notational in nature.

Example 3. *(Uniformly distributed types) When consumers are uniformly distributed, the optimality condition on p_b implies $p_b^*(\gamma_1) = (\frac{\gamma_1}{2})(\theta_o + \theta_a)$. Inserting this into the first period optimality condition yields*

$$1 - \gamma_1 = \gamma_1 \Big[\frac{\theta_o}{\theta_o + \beta(\frac{\theta_o + \theta_a}{2})}\Big] \Rightarrow \gamma_1^* = \frac{(2+\beta)\theta_o + \beta\theta_a}{(4+\beta)\theta_o + \beta\theta_a} > \frac{2+\beta}{4+\beta} > \lambda_f = \frac{1}{2}.$$

Thus, the inclusion of a device to differentiate sales of the accessory introduces an avenue for qualities to affect the equilibrium cutoffs. In contrast to the case without bundling or tracking, consumers must consider two possible advantages of waiting to buy until the second period. Being part of the pool of lower-type buyers not only means you can get the original good more cheaply; now it means you can get the accessory more cheaply, too! For the firm, differentiation creates just another form of self-competition.

The result that $\frac{(2+\beta)\theta_o + \beta\theta_a}{(4+\beta)\theta_o + \beta\theta_a} > \frac{2+\beta}{4+\beta}$ implies that when the firm decides to sell the accesssory to everyone who owns the original, it will attract fewer early buyers under bundling or tracking than without them. It will also sell the original to less consumers in general, although it will sell more of the accessory. When the firm adds the extra incentive for consumers to under-represent their types by offering two different prices for the accessory, it must compensate by becoming more restrictive in its sales of the original.

5.2 Case 2. Some early buyers only buy the original.

When the firm desires not to sell the accessory to all of the early buyers, $p_a^* = \lambda_f \theta_a$. The lowest-type buyer of the original is the one who is indifferent, and therefore has a type characterized by

$$(1+\beta)\gamma_1\theta_o - p_1 = \beta[\gamma_a(\theta_o + \theta_a) - p_b^*] \Rightarrow p_1^*(\gamma_1) = \gamma_1(\theta_o - \beta\theta_a) + \beta p_b^*(\gamma_1).$$

Since p_a^* is now independent of γ_1, the resulting first period maximization problem for the firm is

$$\max_{\gamma_1} \; [1 - F(\gamma_1)][\gamma_1(\theta_o - \beta\theta_a) + \beta p_b^*(\gamma_1)] + \beta\Big[F(\gamma_1) - F\Big(\frac{p_b^*(\gamma_1)}{\theta_o + \theta_a}\Big)\Big]p_b^*(\gamma_1) \Rightarrow$$

$$\max_{\gamma_1} \; [1 - F(\gamma_1)][\gamma_1(\theta_o - \beta\theta_a)] + \beta p_b^*(\gamma_1)\Big[1 - F\Big(\frac{p_b^*(\gamma_1)}{\theta_o + \theta_a}\Big)\Big],$$

for which the corresponding optimality condition when simplified is given by

$$\frac{1 - F(\gamma_1)}{f(\gamma_1)} = \gamma_1\Big[\frac{\theta_o - \beta\theta_a}{\theta_o - \beta\theta_a + \beta p_b^{*\prime}(\gamma_1)}\Big].$$

Example 4. *(Uniformly distributed types) Again, when the consumers' types are uniformly distributed,* $p_b^*(\gamma_1) = (\frac{\gamma_1}{2})(\theta_o + \theta_a)$ *as in the previous case. However, the new optimality condition on* γ_1 *implies that*

$$1 - \gamma_1 = \gamma_1\Big[\frac{\theta_o - \beta\theta_a}{\theta_o - \beta\theta_a + (\frac{\beta}{2})(\theta_o + \theta_a)}\Big] \Rightarrow \gamma_1^* = \frac{(2+\beta)\theta_o - \beta\theta_a}{(4+\beta)\theta_o - \beta\theta_a} < \frac{2+\beta}{4+\beta}.$$

However, recall that this case is only consistent when $\lambda_f > \gamma_1^*$*, since otherwise the firm will still choose to sell the accessory to everyone in the second period. Since* $\lambda_f = \frac{1}{2}$*,* $\gamma^* \geq \lambda_f$ *iff* $\theta_o \geq \theta_a$*. Therefore, when the original good has a higher exogenous quality than the accessory (as would usually be expected given the nature of the model's motivating examples,) case 2 is inconsistent for the firm whose consumers are uniformly distributed.*

When the firm manages to divorce the the advantage gleaned from waiting to buy the original in period 2 from that gleaned by waiting to buy the accessory for a cheaper price, it actually makes waiting seem like a less attractive option than when there was no bundling or tracking. For the consumer who is indifferent, the relative gain from waiting is lessened if she foresees no extra gain from waiting to buy the accessory at the cheaper price. Without tracking or bundling, as well as in Case 1, the indifferent consumer was sure of getting the accessory; now she is sure of not getting it. However, the higher the quality of the accessory, the more burdensome witholding it is for the firm. At some point, the accessory is simply too valuable to refuse to sell to mid-level consumers. Whereas Case 1 represented a tendency to decrese early purchasing of the original good, Case 2 accelerates it.

Tracking and bundling therefore offer the firm some interesting options. If the original good is of relatively high quality, then case 1 will be preferred (and is in fact the only internally-consistent strategy in at least the uniform case). But if the accessory is the more valuable component, the firm may prefer to capitalize off of it by creating two distinct factions of buyers.

References

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